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Rathkjen, Arne

Publication date:
1986

Document Version
Publisher's PDF, also known as Version of record

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In a number of experiments it has been found that in cyclic loading the stress-strain relationship for trabecular bone is as shown in figure 1 (Linde et al., 1985 and 1986). After a few load cycles, order of magnitude 10, the stress-strain curve repeats itself in the loop to the far right in figure 1.

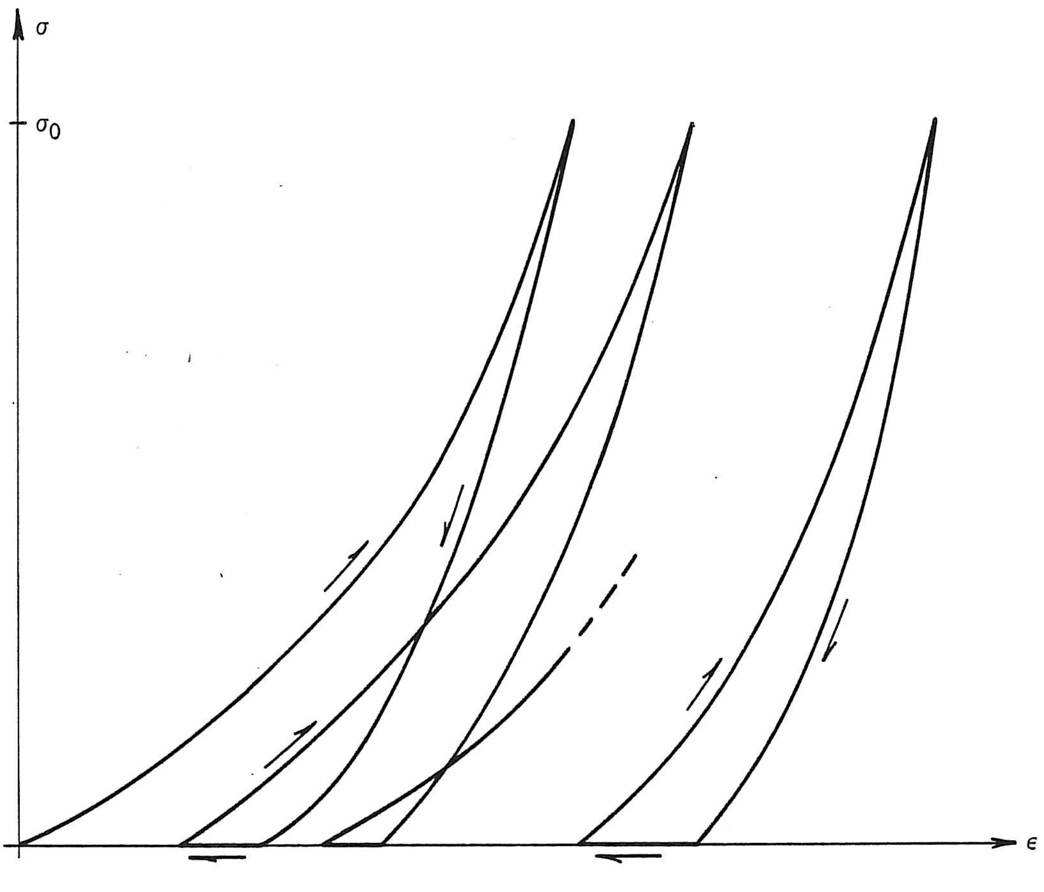

Figure 1. Stress-strain curve for trabecular bone in cyclic loading.

As shown in figure 2 the load cycles consist of 3 phases. In phase 1 the test specimen is loaded in compression with a constant strain rate, $\dot{\epsilon} = k$. The specimen is loaded until the stress reaches a predetermined level, $\sigma = \sigma_0$. In phase 2 the specimen is unloaded with the same constant strain rate, $\dot{\epsilon} = -k$. Unloading takes place until the stress vanishes, $\sigma = 0$. In phase 3 the specimen is allowed to creep freely at $\sigma = 0$ until a predetermined amount of time has elapsed since phase 3 started.

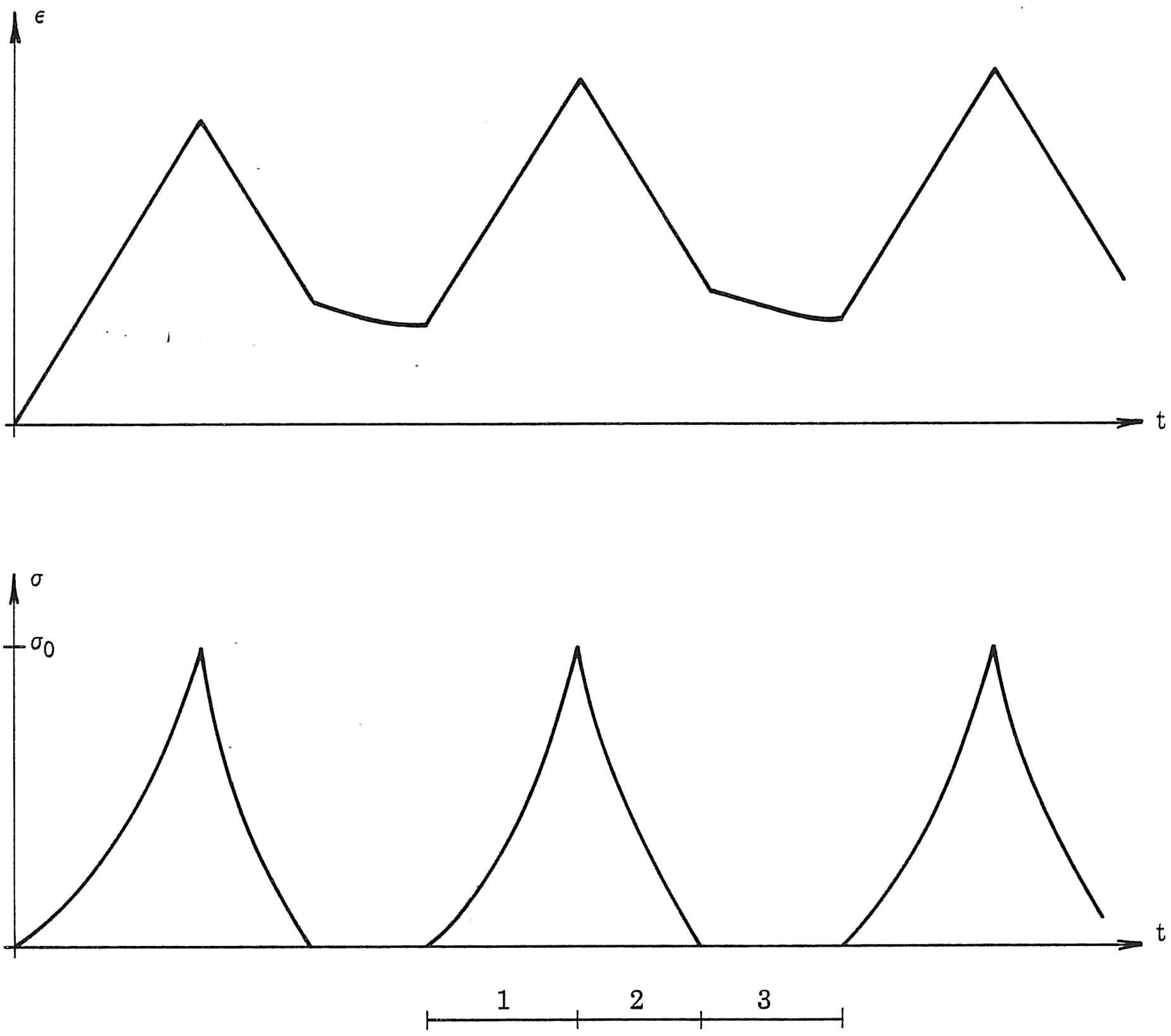

Figure 2. Load cycles, phases 1, 2, and 3.

When a linear, viscoelastic material is loaded as described the stress-strain curve will have the general appearance shown in figure 3. This indicates that the model for trabecular bone has to be a non-linear one.

The non-linear model chosen for investigation is illustrated by a spring and dashpot model in figure 4. It is composed of a Maxwell element in parallel with a non-linear spring. The Maxwell element is characterized by the elasticity E and the viscosity η. When the non-linear spring is not specified the differential equation is

$$\eta\dot{\sigma} + E\sigma = EF(\epsilon) + \eta\dot{F}(\epsilon) + \eta E\dot{\epsilon} \tag{1}$$

or

$$\dot{\sigma} + a\sigma = aF(\epsilon) + \dot{F}(\epsilon) + E\dot{\epsilon} \tag{2}$$

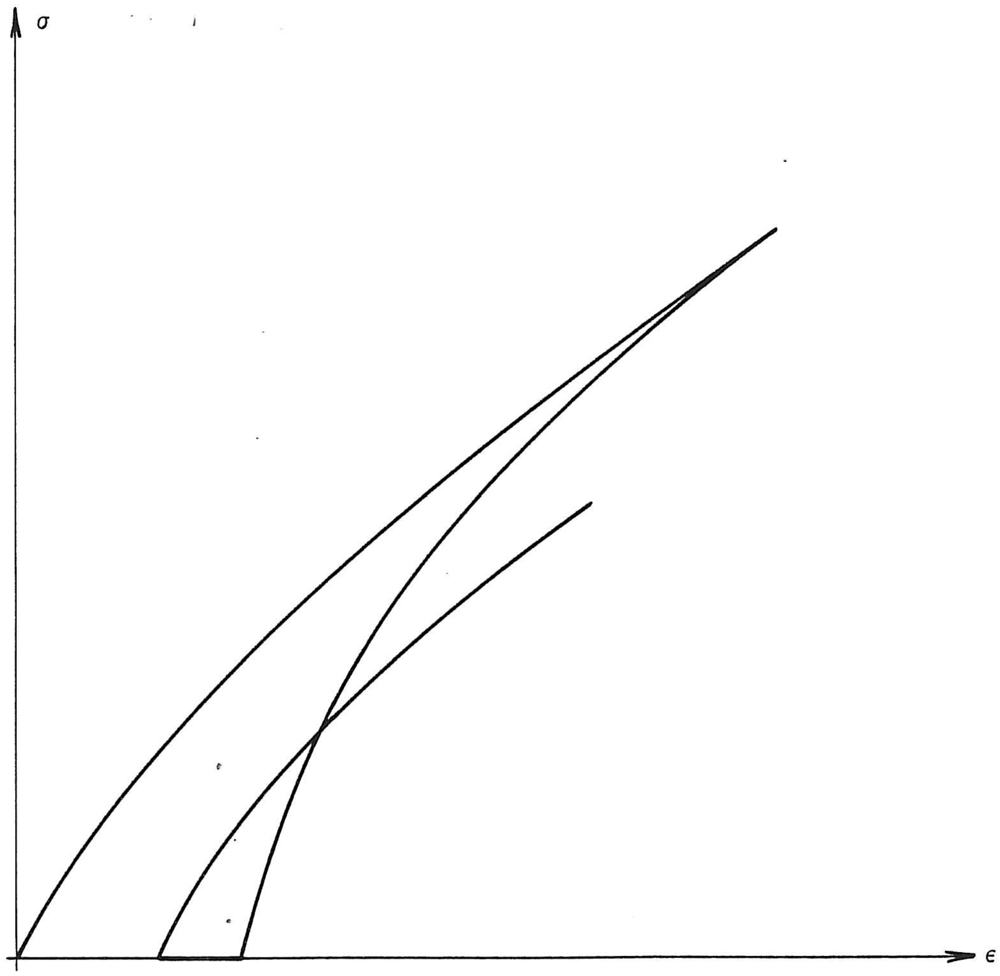

Figure 3. Stress-strain curve for linear, viscoelastic material.

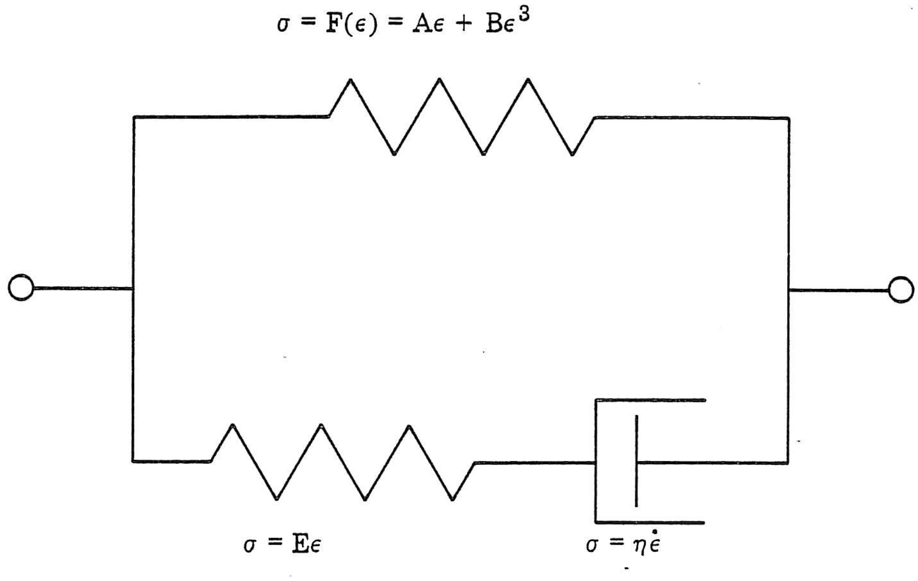

Figure 4. Non-linear, viscoelastic model.

where

$$a = E/\eta \tag{3}$$

For further investigation the non-linear spring is characterized by

$$F(\epsilon) = A\epsilon + B\epsilon^3 \tag{4}$$

where A and B are positive constants. The differential equation now is

$$\dot{\sigma} + a\sigma = aA\epsilon + aB\epsilon^3 + 3B\epsilon^2\dot{\epsilon} + (A + E)\dot{\epsilon} \tag{5}$$

Corresponding to the 3 phases the solutions to the differential equation are:

Phase 1: $\sigma = Bk^3t^3 + Akt + \eta k + C_1 e^{-at}$ (6)

Phase 2: $\sigma = - Bk^3t^3 - Akt - \eta k + C_2 e^{-at}$ (7)

Phase 3: $\ell n(\epsilon(A + B\epsilon^2)^{\alpha_1}) = C_3 - \lambda_1 t$ (8)

where in eq. (8)

$$\alpha_1 = \frac{2A - E}{2(A + E)} \quad , \quad \lambda_1 = \frac{aA}{A + E} \tag{9}$$

The solution (8) does not hold if A = 0. In that case (8) has to be replaced by

$$\ell n\epsilon - \frac{\alpha_2}{\epsilon^2} = C_3 - \lambda_2 t \tag{10}$$

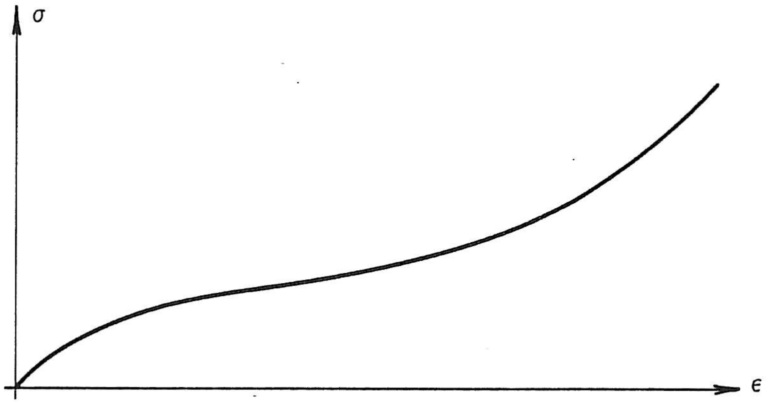

Figure 5. Start of first loading cycle.

where

$$\alpha_2 = \frac{E}{6B} \quad , \quad \lambda_2 = \frac{a}{3} \tag{11}$$

The constants of integration C_1, C_2, and C_3 are determined from initial conditions.

In the first load cycle the theoretical stress-strain curve starts as shown in figure 5. The inflection point is not immediately observed on the experimentally obtained curves, it is assumed to correspond to very small values of ϵ and σ. It is also verified that at the end of phase 2 the stress and the strain have the same sign so that the strain decreases in phase 3 as shown in figure 2.

Algorithms based on equations (6), (7), and (8) are given in the appendix. They were programmed for a digital domputer and simulations with some more or less arbitrarily chosen values of the constants A, B, E, and η show that the theoretical model is able to describe the experimental results. The experiments however, do not supply enough information to determine the constants A, B, E, and η.

In a stress-relation test where the strain and stress vary with time as shown in figure 6 the stress is

$$\text{Phase 1:} \quad \sigma = Bk^3t^3 + Akt + \eta k(1 - e^{-at}) \tag{12}$$

$$\text{Phase 2:} \quad \sigma = B\epsilon_0^3 + A\epsilon_0 + Ce^{-a(t-\epsilon_0/k)} \tag{13}$$

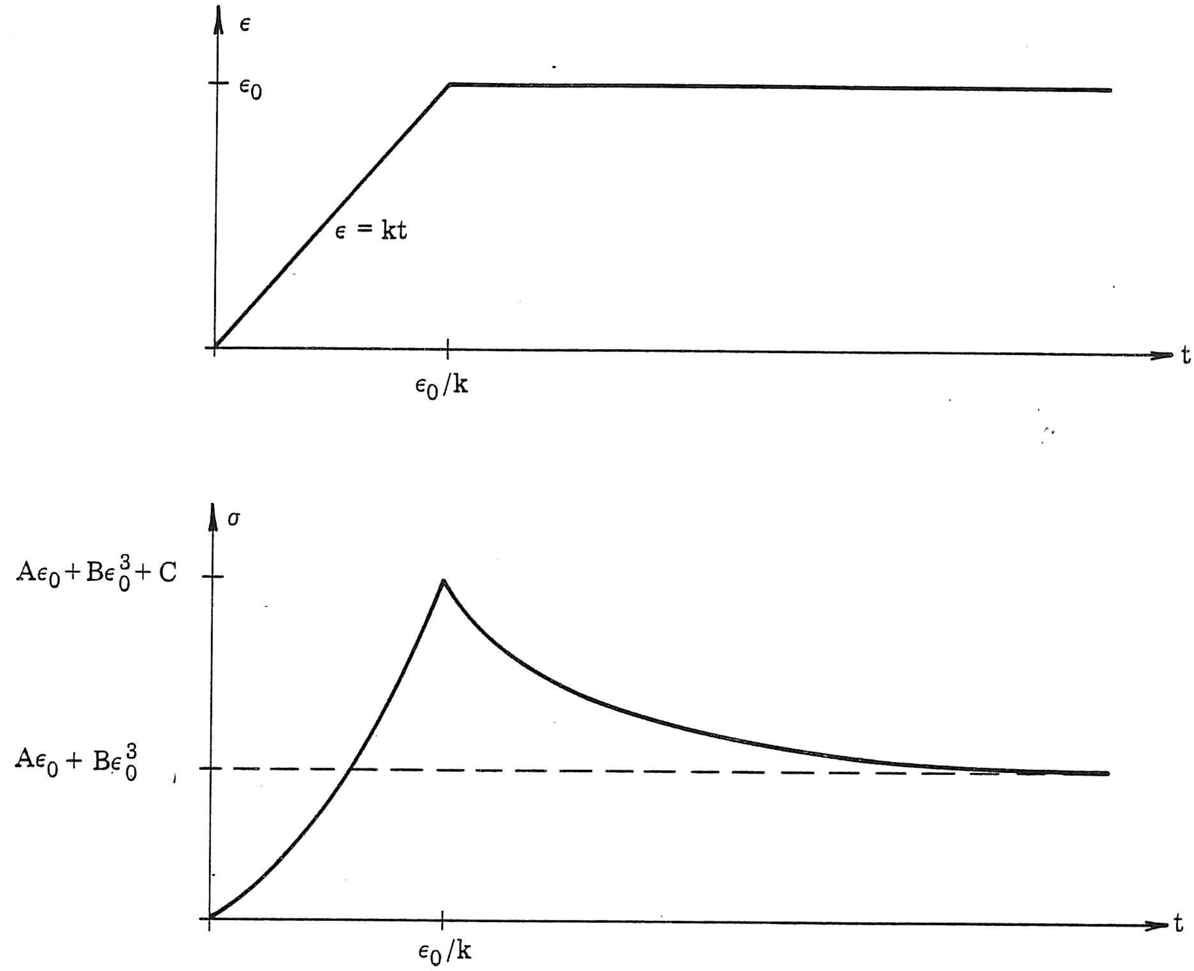

Figure 6. Stress-relaxation test.

where

$$C = \eta k(1 - e^{-a\epsilon_0/k}) \tag{14}$$

Two experiments with separate values of ϵ_0, e.g. ϵ_0 = e and $\epsilon_0 = \alpha$e give

$$\begin{aligned} K_1 &= Ae + Be^3 \\ K_2 &= \alpha Ae + \alpha^3 Be^3 \end{aligned} \tag{15}$$

from which

$$\begin{aligned} A &= \frac{K_2 - \alpha^3 K_1}{\alpha e(1 - \alpha^2)} \\ B &= \frac{\alpha K_1 - K_2}{\alpha e(1 - \alpha^2)} \end{aligned} \tag{16}$$

Also from the same two experiments

$$C_1 = \eta k(1 - e^{-ae/k})$$

$$C_2 = \eta k(1 - e^{-\alpha ae/k}) \qquad (17)$$

giving two equations determining a and η.

The assistance of Tom Brøcher Jakobsen, who did the programming and ran the simulations on the computer, is gratefully acknowledged.

REFERENCES

Linde, F., Hvid, I. & Jensen, N. C. (1985): Material Properties of Cancellous Bone in Repetitive Axial Loading. Engineering in Medicine, 14, 173-177.

Linde, F. & Hvid, I. (1986): Stiffness Behaviour of Trabecular Bone Specimens. Submitted for publication in Journal of Biomechanics.

APPENDIX

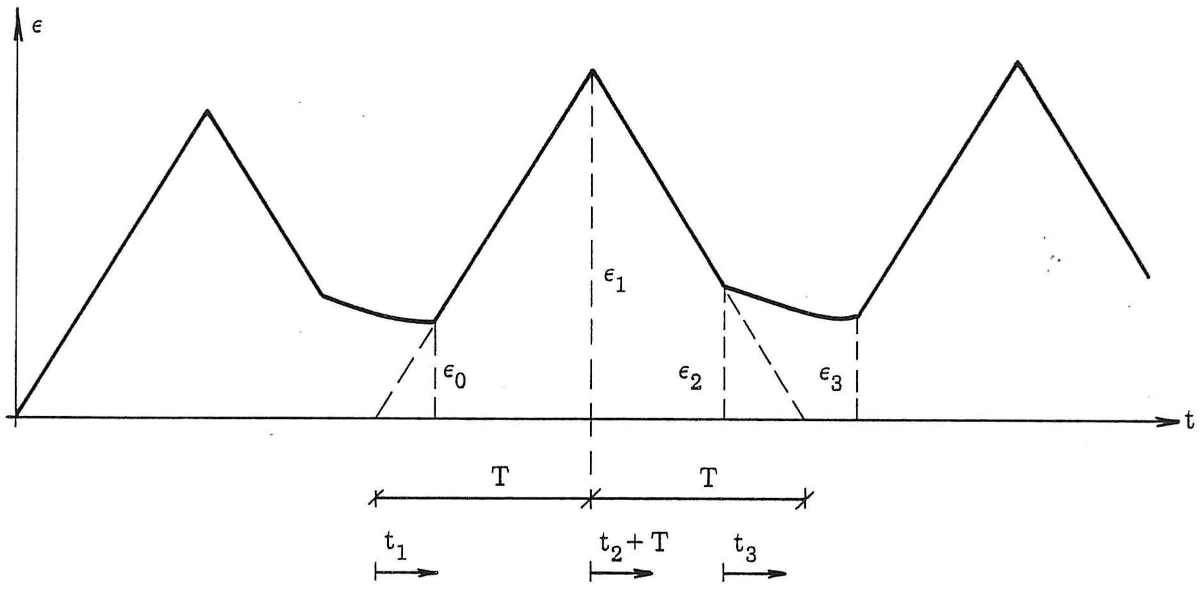

Figure A-1. Notation for algorithms.

In phases 1 and 2 the computations are carried out for a number of time steps, in phase 3 for a number of strain steps. Notation is given in figure A-1.

Phase 1

Start value ϵ_0. Time step Δt_1. Time $t_1 = t_0 + n\Delta t_1$ where $n = 0, 1, 2, \ldots, N_1$ and $t_0 = \epsilon_0/k$. For each value of t_1

$$\sigma = Bk^3t_1^3 + Akt_1 + \eta k + C_1 \exp(-at_1)$$

where $C_1 = -(Bk^3t_0^3 + Akt_0 + \eta k)\exp(at_0)$, is computed until $\sigma \geqslant \sigma_0$ corresponding to $n = N_1$, $T = t_0 + N_1\Delta t_1$ and $\epsilon_1 = kT$.

Phase 2

Start values ϵ_1, T, C_1. Time step Δt_2. Time $t_2 = n\Delta t_2 - T$, $n = 0, 1, 2, \ldots, N_2$. For each value of t_2

$$\sigma = -Bk^3t_2^3 - Akt_2 - \eta k + C_2 \exp(-at_2)$$

where $C_2 = (2\eta k + C_1 \exp(-aT))\exp(-aT)$, is computed until $\sigma \leqslant 0$ corresponding to $n = N_2$, $S = N_2\Delta t_2 - T$ and $\epsilon_2 = -kS$.

Phase 3

Start values ϵ_2, ϵ_0. Strain step $\Delta\epsilon = (\epsilon_2 - \epsilon_0)/N$, where N is a predetermined number of steps. Strain $\epsilon = \epsilon_2 - n\Delta\epsilon$, $n = 0, 1, 2, \ldots, N_3$. For each value of ϵ

$$t_3 = (C_3 - \ell n(\epsilon(A + B\epsilon^2)^{\alpha_1}))/\lambda_1$$

where $C_3 = \ell n(\epsilon_2(A + B\epsilon_2^2)^{\alpha_1})$ is computed. α_1 and λ_1 are given by (9). Computations are repeated until $t_3 \geqslant R$, where R is the given time interval. The corresponding value of ϵ is $\epsilon_3 = \epsilon_2 - N_3\Delta\epsilon$.